Mathematical Enrichment   Feb 7^{th} 2015

Kevin Hutchinson , Inequalities.

$A < B$ A is less than B

$\Updownarrow$

$B > A$ B is greater than A

$A \leq B$ A is less than or equal to B.

$B \geq A$ B is greater than . . . . . A.

Eg. $a, b > 0$

$$A = \frac{1}{a} + \frac{1}{b} \qquad \frac{1}{a+b} = B$$

Note $a+b > a$
$a+b > b$

$$\Rightarrow \quad \frac{1}{a+b} < \frac{1}{a}$$

$$\text{and} \quad \frac{1}{a+b} < \frac{1}{b}$$

It follows that

$$\frac{1}{a} + \frac{1}{b} > \frac{1}{a+b} + \frac{1}{a+b}$$

Q: $\frac{1}{a} + \frac{1}{b} \qquad \frac{1}{a+b} + \frac{1}{a+b} + \frac{1}{a+b} + \frac{1}{a+b}$

$$\frac{1}{a} + \frac{1}{b} \quad ? \quad \frac{4}{a+b}$$

1

$$\frac{a+b}{ab} \geqslant \frac{4}{a+b} \quad ?$$

$$\Longleftrightarrow \quad \boxed{(a+b)^2 \geqslant 4ab} \quad ?$$

since $a,b>0$

$$a^2+2ab+b^2 \geqslant 4ab \quad ?$$

$$\Longleftrightarrow \quad a^2-2ab+b^2 \geqslant 0$$

$$\Longleftrightarrow \quad (a-b)^2 \geqslant 0$$

true since $\boxed{x^2 \geqslant 0}$

equality only occurs when $a-b=0$ i.e. when $a=b$

Basic principle One way to prove $A \geqslant B$

i.e. $A-B \geqslant 0$

is to show that $A-B$ is a square.

Example: a_1, a_2, b_1, b_2 any numbers.

$$\underbrace{(a_1^2+a_2^2)\cdot(b_1^2+b_2^2)}_{A=} \geqslant \underbrace{(a_1b_1+a_2b_2)^2}_{B}$$

"Cauchy's inequality"

$$A-B=C^2 \qquad \text{what's } C?$$

$$A-B = (a_1^2+a_2^2)(b_1^2+b_2^2) - (a_1b_1+a_2b_2)^2$$

$$= \cancel{a_1^2b_1^2} + a_1^2b_2^2 + a_2^2b_1^2 + \cancel{a_2^2b_2^2} - \cancel{a_1^2b_1^2} - 2a_1b_1a_2b_2 - \cancel{a_2^2b_2^2}$$

$$= a_1^2b_2^2 - 2a_1b_2a_2b_1 + a_2^2b_1^2$$

$$= (a_1b_2-a_2b_1)^2. \quad \text{So } C = \pm(a_1b_2-a_2b_1).$$

Exercise $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$ $\boxed{\text{Case } n=3}$

Show

$$(a_1^2+a_2^2+a_3^2)\cdot(b_1^2+b_2^2+b_3^2) \geqslant (a_1b_1+a_2b_2+a_3b_3)^2$$

Back to $n=2$

$$(a_1^2+a_2^2)(b_1^2+b_2^2) \geqslant (a_1b_1+a_2b_2)^2$$

When does equality occur?

We saw that difference is $(a_1b_2-a_2b_1)^2 \geqslant 0$

equality only if $a_1b_2-a_2b_1=0$

only if $a_1b_2=a_2b_1$

only if $\frac{a_1}{b_1}=\frac{a_2}{b_2}$ $(b_1, b_2 \neq 0)$.

We showed that

$$(a+b)^2 \geqslant 4ab \quad \text{if } a,b>0$$

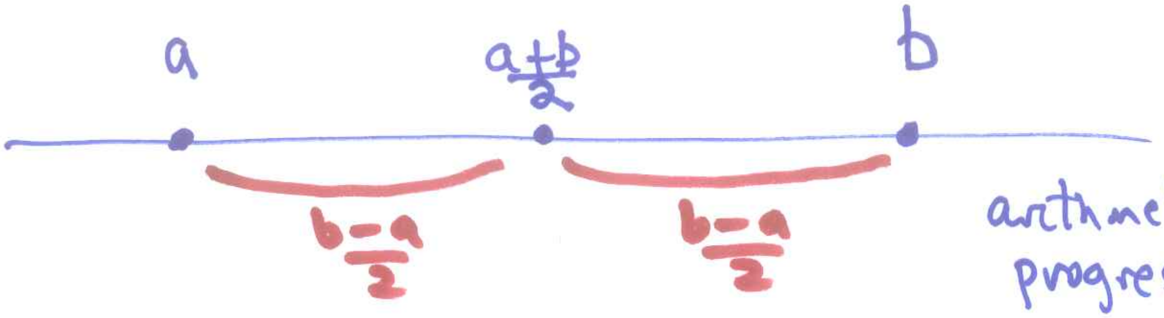
$$a+b \geqslant 2\sqrt{ab}$$

i.e. $\boxed{\frac{a+b}{2} \geqslant \sqrt{ab}}$ if $a,b>0$

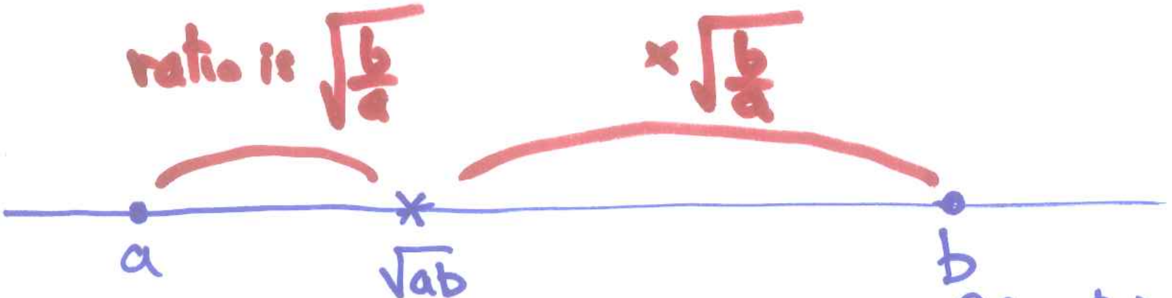
equality occurs if and only if $\boxed{a=b}$

arithmetic mean (average) of a and b

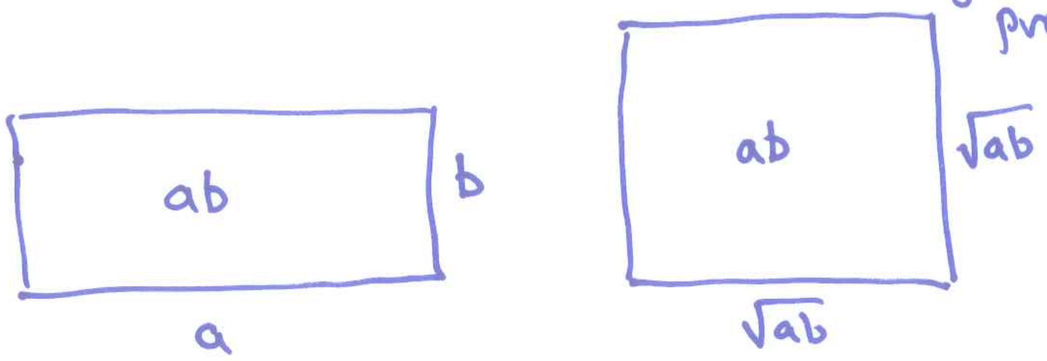
geometric mean



arithmetic progression.



geometric progession.



$$\frac{a+b}{2} \geq \sqrt{ab} \qquad \text{if } a, b \geq 0$$

A·M – G·M inequality

Example 2 positive numbers sum to 50.
What is the maximum possible value of their product?

We have $ab \leq \left(\frac{a+b}{2}\right)^2 = \left(\frac{50}{2}\right)^2 = 25^2$

and equality occurs when $a = b = 25$

What is the largest possible value of xy $\{x, y > 0\}$ if $5x + 3y = 10$?

$$a = 5x, \quad b = 3y$$

$$ab \leq \left(\frac{a+b}{2}\right)^2$$

So $$5x \cdot 3y \leq \left(\frac{10}{2}\right)^2 \leq 5^2$$

$$15xy \leq 5^2$$

$$\therefore \quad xy \leq \frac{5}{3}$$

with equality when $a = b$ i.e. $5x = 3y$

i.e. $y = \frac{5}{3}x$.

So $10 = 5x + 3y = 5x + 5x \Rightarrow \boxed{x = 1, \; y = 5/3.}$

General.

A.M-GM inequality

Given $a_1, a_2, \ldots, a_n \geq 0$

$$\frac{a_1 + \cdots + a_n}{n} \geq (a_1 a_2 \cdots a_n)^{1/n} = \sqrt[n]{a_1 a_2 \cdots a_n}.$$

Arithmetic Mean

Geometric Mean

with equality occurring if and only if

$$a_1 = a_2 = \cdots = a_n$$

$$\Longleftrightarrow \quad a_1 + \cdots + a_n \geq n \sqrt[n]{a_1 \cdot \cdots \cdot a_n}$$

$$\Longleftrightarrow \quad a_1 a_2 \cdots a_n \leq \left(\frac{a_1 + \cdots + a_n}{n}\right)^n$$

We've shown the case $n=2$: $\left(\frac{a_1+a_2}{2}\right) \leq \sqrt{a_1a_2}$

$\underline{n=4.}$

- To prove

$$a_1a_2a_3a_4 \leq \left(\frac{a_1+a_2+a_3+a_4}{4}\right)^4$$

Proof:

$$a_1a_2a_3a_4 = (a_1a_2)\cdot(a_3a_4)$$

$$\underset{\substack{\text{AM-GM} \\ n=2.}}{\leq} \left(\frac{a_1+a_2}{2}\right)^2 \cdot \left(\frac{a_3+a_4}{2}\right)^2 = \left[\left(\frac{a_1+a_2}{2}\right)\left(\frac{a_3+a_4}{2}\right)\right]^2$$

AM-GM

$$\underset{\substack{\text{AM-GM} \\ n=2}}{\leq} \left(\left(\frac{\frac{a_1+a_2}{2} + \frac{a_3+a_4}{2}}{2}\right)^2\right)^2$$

$$\parallel$$

$$\left(\frac{a_1+a_2+a_3+a_4}{4}\right)^4 \quad \checkmark.$$

equality only if $\underbrace{a_1 = a_2 \text{ and } a_3 = a_4 \text{ and } \frac{a_1+a_2}{2} = \frac{a_3+a_4}{2}}$

$$\Longleftrightarrow a_1 = a_2 = a_3 = a_4.$$

In the same way we deduce the case $n=8$ from the case $n=4$. $(\underbrace{a_1 \cdots a_4})(\underbrace{a_5 \cdots a_8}) \cdots$ etc

case $n \Rightarrow$ case $2n$ by this argument.

$n=2,\ n=4,\ n=8,\ n=16,\ n=32, \cdots\cdots$

$n = 2^k$ $\underline{\underline{\text{ok}}}$

(by induction on k).

Lets show that if we know AM-GM inequality for $n+1$ numbers, we can deduce it for n numbers.

Know it for <u>any</u> $n+1$ numbers.

Suppose given n numbers $a_1, a_2, \ldots, a_n \geq 0$.

Let $A = \dfrac{a_1 + \cdots + a_n}{n}$ $(= a_{n+1})$

We know AM-GM for $a_1; a_2, \ldots, a_n, A$

$$\therefore \quad \frac{a_1 + a_2 + \cdots + a_n + A}{n+1} \leq (a_1 a_2 \cdots a_n \cdot A)^{\frac{1}{n+1}}$$

$$\|$$

$$\frac{a_1 + a_2 + \cdots + a_n + \dfrac{(a_1 + \cdots + a_n)}{n}}{n+1} = \frac{\dfrac{\cancel{(n+1)}(a_1 + \cdots + a_n)}{n}}{\cancel{n+1}}$$

$$= \frac{a_1 + a_2 + \cdots + a_n}{n} = A$$

So $\quad A \leq (a_1 a_2 \cdots a_n)^{\frac{1}{n+1}} \cdot A^{1/n+1}$

So $\quad A^{1-\frac{1}{n+1}} \leq (a_1 a_2 \cdots a_n)^{\frac{1}{n+1}}$

ie. $\quad A^{n/n+1} \leq (a_1 a_2 \cdots a_n)^{1/n+1}$

raise both sides to the power $\frac{n+1}{n}$

$$A \leq (a_1 \cdots a_n)^{1/n}$$

$$\shortparallel$$

$$\frac{a_1 + \cdots + a_n}{n}$$

Done

Example Find the least value of

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \qquad \text{for } x, y, z > 0$$

$$a_1 + a_2 + a_3 \geqslant 3\sqrt[3]{a_1 \cdot a_2 \cdot a_3}$$

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geqslant 3\sqrt[3]{1} = 3$$

~~when~~ equality if $\frac{x}{y} = \frac{y}{z} = \frac{z}{x}$ $(\Leftrightarrow x = y = z)$

Exercises 1. Find the smallest value of

$$\frac{3}{x} + \frac{4}{y} + xy \qquad x, y > 0$$

2. Find the minimum of

$$x^2 + \frac{4}{x} \qquad \text{for all } x > 0$$

(without calculus!)

3. Show that

$$(a+b)(b+c)(c+a) \geq 8abc \quad \text{if } a,b,c > 0$$

4. Show that

$$x^2 + y^2 + z^2 \geq xy + yz + zx \quad , \quad x,y,z \geq 0$$

5. Show that $x^3 + y^3 + z^3 \geq 3xyz$

if $x, y, z \geq 0$